

Example 3 :

In a school, there are 1000 students, out of which 430 are girls. It is known that out of 430, 10% of the girls study in class XII. What is the probability that a student chosen randomly studies in Class XII given that the chosen student is a girl?

Solution:

Let A denote the event that a student chosen randomly studies in Class XII and B be the event that the randomly chosen student is a girl. We have to find $P(A|B)$.

Now

$$P(B) = \frac{430}{1000} = 0.43 \quad \text{and} \quad P(A \cap B) = \frac{43}{1000} = 0.043 \quad (\text{Why?})$$

Then

$$P(A|B) = \frac{P(A \cap B)}{P(B)} = \frac{0.043}{0.43} = 0.1$$

Example 4 :

A die is thrown twice and the sum of the numbers appearing is observed to be 6. What is the conditional probability that the number 4 has appeared at least once?

Solution:

Let E be the event that 'number 4 appears at least once' and F be the event that 'the sum of the numbers appearing is 6'.

Then $E = \{(4, 1), (4, 2), (4, 3), (4, 4), (4, 5), (4, 6), (1, 4), (2, 4), (3, 4), (5, 4)\}$ and $F = \{(1, 5), (2, 4), (3, 3), (4, 2), (5, 1)\}$

We have

$$P(E) = \frac{11}{36} \quad \text{and} \quad P(F) = \frac{5}{36}$$

Also

$$E \cap F = \{(2, 4), (4, 2)\}$$

Cont ...

Therefore

$$P(E \cap F) = \frac{2}{36}$$

Hence, the required probability

$$P(E|F) = \frac{P(E \cap F)}{P(F)} = \frac{\frac{2}{36}}{\frac{5}{36}} = \frac{2}{5}$$

Odds of an event:-

The odds of an event A are defined by

$$\frac{P(A)}{P(A^c)} = \frac{P(A)}{1 - P(A)}$$

i.e. the odds of an event A tells how much more likely it is that the event A occurs than it ~~is~~ does not occur.

eg if $P(A) = \frac{2}{3}$, then $P(A^c) = 1 - \frac{2}{3} = \frac{1}{3}$

then $P(A) = 2 P(A^c) \Rightarrow \frac{P(A)}{P(A^c)} = 2$

so the odds are 2.

• If the odds are equal to α , then it is common to say that the odds are " α to 1" in favor of the hypothesis.

Model

Let H be true with probability $P(H)$.
And suppose the conditional probability, given the evidence E , that H is true and that H is not true are respectively given by

$$P(H|E) = \frac{P(E|H) P(H)}{P(E)}$$

$$P(H^c|E) = \frac{P(E|H^c) P(H^c)}{P(E)}$$

∴ the new odds after the evidence E has been introduced are,

$$\frac{P(H|E)}{P(H^c|E)} = \frac{P(H)}{P(H^c)} \frac{P(E|H)}{P(E|H^c)}$$

i.e. the new value of the odds of H is the old value multiplied by the ratio of the conditional probability of the new evidence given that H is true to the conditional probability of the new evidence given that H is not true.

Thus the probability of H increases whenever the new evidence is more likely when H is true than when it is false. Similarly, the odds decrease whenever the new evidence is more likely when H is false than when it is true.

→ Suppose that $F_1, F_2, \dots, F_n$ are mutually exclusive events such that

$$\bigcup_{i=1}^n F_i = S$$

In other words exactly one of the events $F_1, F_2, \dots, F_n$ must occur. By writing

$$E = \bigcup_{i=1}^n E F_i$$

and using the fact that the events $E F_i$, $i=1, \dots, n$ are mutually exclusive, we get

$$P(E) = \sum_{i=1}^n P(E F_i) \\ = \sum_{i=1}^n P(E|F_i)P(F_i) \quad \text{--- (2)}$$

Eq (2) referred to as the law of total probability. and states that $P(E)$ is equal to a weighted average of $P(E|F_i)$, each term being weighted by the probability of the event on which it is conditioned.

Proposition 3.1 (Bayes's formula) / Bayes's Theorem

If $F_1, F_2, \dots, F_n$ are mutually disjoint events with $P(F_i) > 0$, ($i=1, 2, \dots, n$) then for any arbitrary event E which is a subset of $\bigcup_{i=1}^n F_i$ such that $P(E) > 0$, we have

$$P(F_j | E) = \frac{P(E F_j)}{P(E)} = \frac{P(E|F_j)P(F_j)}{\sum_{i=1}^n P(E|F_i)P(F_i)} \quad (1)$$

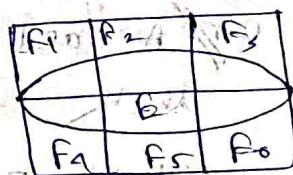
Eq (1) is known as Bayes's formula.

Since $E \subseteq \bigcup_{i=1}^n F_i$ we have

$$E = E \cap \left(\bigcup_{i=1}^n F_i \right) = \bigcup_{i=1}^n E F_i$$

Since $E F_i \subseteq F_i$, $i=1, 2, \dots, n$ are mutually disjoint events, we have by addition that

$$P(E) = P\left\{ \bigcup_{i=1}^n E F_i \right\} = \sum_{i=1}^n P(E F_i) \\ = \sum_{i=1}^n P(E|F_i)P(F_i)$$



Also we have

$$P(EF_j) = P(E) P(E|F_j)$$

$$\Rightarrow P(F_j|E) = \frac{P(EF_j)}{P(E)}$$

$$= \frac{P(F_j) P(E|F_j)}{\sum_{i=1}^n P(E|F_i) P(F_i)}$$

□

Independent Events:-

Since $P(E|F) = \frac{P(EF)}{P(F)}$

E is independent of F if

$$P(EF) = P(E) P(F) \quad (1)$$

$$\Rightarrow P(E|F) = \frac{P(E) P(F)}{P(F)} = P(E)$$

Eq (1) is symmetric in E & F shows that whenever E is independent of F , F is also independent of E .

→ Two events E & F are said to be independent if eq (1) holds.

→ Two events E & F that are not independent are said to be dependent.

eg // A card is selected at random from an ordinary deck of 52 playing cards.

Let E = the selected card is an ace.

F = it is a spade.

now E & F are independent.

because $P(E) = \frac{4}{52}$, $P(F) = \frac{13}{52}$

$$P(EF) = P(E) \cdot P(F) = \frac{1}{52} \quad (EF = \text{ace of spades})$$

eg // Two coin are flipped, and all 4 outcomes are assumed to be equally likely.

E = the first coin lands on head = $\{(h, h), (h, t)\}$

F = the second lands on tail.

$$= \{(h, t), (t, t)\}$$

$$P(E) = \frac{1}{2}, \quad P(F) = \frac{1}{2}$$

$$P(EF) = \{(h, t)\} = \frac{1}{4} = P(E) \cdot P(F)$$

here E & F are independent.

Proposition 4.1

If E & F are independent, then so are E & F^c .

pf Let E & F are independent.

$$\text{Since } E = EF \cup EF^c$$

& EF & EF^c are obviously mutually exclusive

$$\text{we have } P(E) = P(EF) + P(EF^c)$$

$$\Rightarrow P(E) = P(E)P(F) + P(EF^c)$$

$$\Rightarrow P(EF^c) = P(E)[1 - P(F)]$$

$$= P(E)P(F^c)$$

So E & F^c are independent.

eg - [If E is independent of F & F is also independent of G , then it is not necessary that it is independent of $F \cap G$]

~~Let E = event~~ Let two fair dice are thrown

E = event that the sum of die is 7.

F = ~~sum~~ 1st die is 4.

G = 2nd die is 3.

$$\text{now } P(E) = \frac{1}{6}, \quad P(F) = \frac{1}{6}$$

$$P(EF) = P\{(4, 3)\} = \frac{1}{36}$$

$$P(E)P(F) = \frac{1}{6} \cdot \frac{1}{6} = \frac{1}{36}$$

So E is independent of F .

Similarly

E is also independent of G ,
but clearly E is not independent of $F \cap G$.

$$\therefore P(E | F \cap G) = 1$$

Defn.

Three events E, F , & G are said to be independent
iff $P(E \cap F \cap G) = P(E) P(F) P(G)$

$$P(E \cap F) = P(E) P(F)$$

$$P(E \cap G) = P(E) P(G)$$

$$P(F \cap G) = P(F) P(G)$$

NOTE

If E, F & G are independent, then E will be
independent of any event formed from F & G .
 E is independent of $F \cup G$.

$$\begin{aligned} P[E \cap (F \cup G)] &= P(E \cap F \cup E \cap G) \\ &= P(E \cap F) + P(E \cap G) - P(E \cap F \cap G) \\ &= P(E) P(F) + P(E) P(G) - P(E) P(F \cap G) \\ &= P(E) [P(F) + P(G) - P(F \cap G)] \\ &= P(E) P(F \cup G) \end{aligned}$$

→ The events $E_1, E_2, \dots, E_n$ are said to be
independent if for every subset $E_1, E_2, \dots, E_r$,
 $r \leq n$ of these events,

$$P(E_1 \cap E_2 \cap \dots \cap E_r) = P(E_1) P(E_2) \dots P(E_r)$$

→ An infinite set of events is said to be
independent if every finite subset of
those events is independent.